

Non-equilibrium dynamics of Ising models with decoherence: an exact solution

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The interplay between interactions and decoherence in many-body systems is of fundamental importance in quantum physics: Decoherence can degrade correlations, but can also give rise to a variety of rich dynamical and steady-state behaviors. We obtain an exact analytic solution for the non-equilibrium dynamics of Ising models with arbitrary interactions and subject to the most general form of local Markovian decoherence. Our solution shows that decoherence affects the relaxation of observables more than predicted by single-particle considerations. It also reveals a dynamical phase transition, specifically a Hopf bifurcation, which is absent at the single-particle level. These calculations are applicable to ongoing quantum information and emulation efforts using a variety of atomic, molecular, optical, and solid-state systems.

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Understanding strongly correlated quantum systems in the presence of decoherence is a fundamental challenge in modern physics. While decoherence generally tends to degrade correlations, it is now widely appreciated that it can also give rise to many-body physics not possible with strictly coherent dynamics [1–6], and can be used explicitly for the creation of entanglement[7–10]. Regardless of whether one’s intention is to minimize or to harness decoherence, determining its effect on interacting many-body systems is central to quantum simulation[11], quantum information [12], and quantum metrology [13]. So far, this understanding has been hindered by the computational complexity of numerical techniques for open systems and the scarcity of exact analytic solutions. Exact solutions for dynamics of interacting quantum systems in dimensions greater than one are rare even in the absence of decoherence, and to our knowledge no such solutions have been obtained in the presence of local decoherence.

The central result of this manuscript is an exact solution, Eqs. (9-11), for the time-dependence of all two-spin correlation functions in a system of spins interacting via arbitrary Ising couplings, and subject to the most general form of local Markovian decoherence allowed by nature [14]. Our solution is applicable to a broad range of important quantum systems, including trapped ions [15–17], polar molecules [18, 19], Rydberg atoms [20, 21], neutral atoms in optical cavities [22, 23], optical lattice clocks [24], superconducting qubits[25], quantum dots[26], and nitrogen vacancy centers[27]. Here we apply our solution to trapped ion experiments because: (1) the relative importance of decoherence and coherently driven quantum correlations is controllable, (2) the tunable long-ranged interactions are generically frustrated, making large-scale numerical simulations impractical, and (3) these experiments are the most developed, with non-equilibrium dynamics already being explored [15].

Outline of the calculation.—We consider far-from-

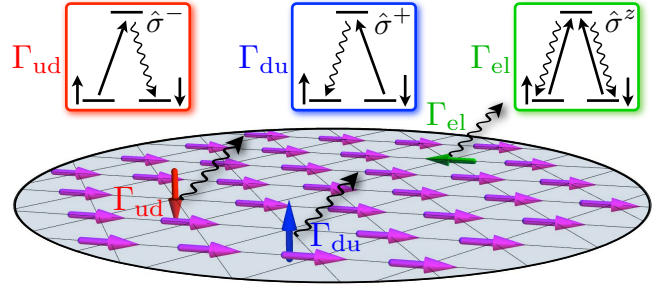


FIG. 1: (Color online) Schematic illustration of the various decoherence processes. The figure shows a lattice of spins initialized to point in a direction orthogonal to the z -axis. Longitudinal (T_1) spin relaxation with rates Γ_{ud} , Γ_{du} (red and blue spin, respectively), and dephasing (T_2) with rate Γ_{el} (green spin) are shown for three different spins. In atomic systems, one way this decoherence can arise is due to spontaneous emission from an excited level (top panels) [28].

equilibrium dynamics of a long-ranged Ising Hamiltonian

$$\mathcal{H} = \frac{1}{\mathcal{N}} \sum_{i < j} J_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z. \quad (1)$$

Here $\hat{\sigma}^a$ ($a = x, y, z$) are Pauli matrices, \mathcal{N} is the total number of spins, and Latin letters from the middle of the alphabet are site indices. Our results are valid for arbitrary J_{ij} , but given the relevance to numerous experiments we will sometimes consider power law couplings $J_{ij} = J|\mathbf{r}_i - \mathbf{r}_j|^{-\zeta}$, where \mathbf{r}_i is the position of the i^{th} spin in lattice units ($\zeta = 3$ for polar molecules, $\zeta = 6$ for Rydberg atoms, and $0 < \zeta < 3$ for trapped ions). In the presence of local decoherence, the most general Markovian dynamics of the system reduced density matrix obeys a master equation [37]

$$\hbar \dot{\rho} = -i \left(\mathcal{H}_{\text{eff}} \rho - \rho \mathcal{H}_{\text{eff}}^\dagger \right) + \mathcal{D}(\rho). \quad (2)$$

The effective Hamiltonian \mathcal{H}_{eff} and dissipator \mathcal{D} have contributions from all jump operators $\mathcal{J} \in$

$$\{\hat{\sigma}_j^- \sqrt{\Gamma_{\text{ud}}/2}, \hat{\sigma}_j^+ \sqrt{\Gamma_{\text{du}}/2}, \hat{\sigma}_j^z \sqrt{\Gamma_{\text{el}}/8} : 1 \leq j \leq \mathcal{N}\},$$

$$\mathcal{H}_{\text{eff}} = \mathcal{H} - i \sum_{\text{all } \mathcal{J}} \mathcal{J}^\dagger \mathcal{J}, \quad \mathcal{D}(\rho) = 2 \sum_{\text{all } \mathcal{J}} \mathcal{J} \rho \mathcal{J}^\dagger, \quad (3)$$

where $\hat{\sigma}_j^\pm = \frac{1}{2}(\hat{\sigma}_j^x \pm i\hat{\sigma}_j^y)$. The jump operators $\hat{\sigma}^-$, $\hat{\sigma}^+$, and $\hat{\sigma}^z$ give rise to spontaneous de-excitation, spontaneous excitation, and elastic dephasing, respectively (see Fig. 1). We refer to the spin-changing processes ($\hat{\sigma}^\pm$) as Raman decoherence, and the spin-preserving processes ($\hat{\sigma}^z$) as Rayleigh decoherence. In what follows we assume an initially pure and uncorrelated density operator $\rho(0) = |\psi(0)\rangle\langle\psi(0)|$, with $|\psi(0)\rangle = \bigotimes_j \sum_{\sigma_j} f_j(\sigma_j) |\sigma_j\rangle$. Here $\sigma_j = \pm 1$ are the eigenvalues of $\hat{\sigma}_j^z$, $f_j(1) = \cos(\theta_j/2)e^{i\varphi_j/2}$, and $f_j(-1) = \sin(\theta_j/2)e^{-i\varphi_j/2}$, for arbitrary θ_j and φ_j .

Our approach to the solution of Eq. (2) for the chosen initial conditions and arbitrary $\{\Gamma_{\text{ud}}, \Gamma_{\text{du}}, \Gamma_{\text{el}}\}$ is based on the quantum trajectories method [29], in which $|\psi(0)\rangle$ is time-evolved with the effective Hamiltonian

$$\mathcal{H}_{\text{eff}} = \mathcal{H} - \frac{i}{2} \sum_j \left(\frac{\Gamma_r}{2} + \frac{\Gamma_{\text{el}}}{4} + 2\gamma\hat{\sigma}_j^z \right), \quad (4)$$

and the dynamics are interspersed with stochastic applications of the jump operators. In Eq. (4) we have defined $\Gamma_r = \Gamma_{\text{ud}} + \Gamma_{\text{du}}$ and $\gamma = \frac{1}{4}(\Gamma_{\text{ud}} - \Gamma_{\text{du}})$. According to the standard prescription [29], a particular trajectory consists of a set of jump times $\{t_1, t_2, \dots\}$, which are selected by equating the norm of the wave-function to a random number uniformly distributed between 0 and 1. Choosing which jump operator to apply at each time requires calculating all expectation values $\langle \mathcal{J}^\dagger \mathcal{J} \rangle$. Because \mathcal{H} is Hermitian and commutes with all products $\mathcal{J}^\dagger \mathcal{J}$, it has no effect on the selection of the jumps, which can therefore be obtained for each spin independently (since the anti-Hermitian part of \mathcal{H}_{eff} does not couple different spins). With the jump times and jump operators in hand, we define a string of n_j (time-labeled) jump operators on site j as $\hat{\mathcal{Q}}_j = \mathcal{J}_j^{n_j}(t_j^{n_j}) \times \dots \times \mathcal{J}_j^{n_j}(t_j^1)$. The time evolution of the wavefunction along a trajectory is then

$$|\psi(t)\rangle = \mathcal{T}(e^{-i\mathcal{H}_{\text{eff}}t} \prod_j \hat{\mathcal{Q}}_j) |\psi(0)\rangle, \quad (5)$$

where the time-ordering operator \mathcal{T} enforces that the jump operators are interspersed in the time evolution according to their time labels.

The time-ordering of the Rayleigh jumps can be ignored: Because $[\hat{\sigma}_j^z, \mathcal{H}_{\text{eff}}] = 0$ and $[\hat{\sigma}_j^z, \hat{\sigma}_j^\pm] = \pm 2\hat{\sigma}_j^\pm$, all Rayleigh jumps can be evaluated at $t = 0$ (their commutation with Raman jumps affects only the overall sign of the wavefunction). To the contrary, the Raman jumps do not commute with \mathcal{H}_{eff} , and their time ordering cannot be so easily accounted for. However, imagine that spin j undergoes a single Raman jump, created by applying $\hat{\sigma}_j^\pm$ at time t . This jump operator not only flips spin j into

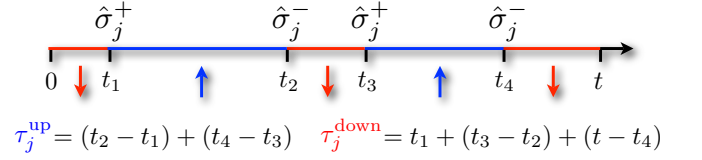


FIG. 2: (Color online) A series of Raman flips of the spin on site j can be formally accounted for as a magnetic field of strength $2J_{jk}/\mathcal{N}$ acting for a period of time $\tau_j^{\text{up}} - \tau_j^{\text{down}}$. In the notation defined below, this series of jumps is represented by the operator $\hat{\mathcal{Q}}_j \propto \hat{\sigma}_j^+(t_1)\hat{\sigma}_j^-(t_2)\hat{\sigma}_j^+(t_3)\hat{\sigma}_j^-(t_4)$.

the up position, but also removes all parts of the wavefunction in which spin j pointed up immediately before time t , and up after time t . Since spin j is always in an eigenstate of $\hat{\sigma}_j^z$, it is a spectator to the Ising dynamics, but it does influence the other spins via the Ising coupling J_{jk} ; formally it acts on spin k as an inhomogeneous magnetic field of strength $2J_{jk}/\mathcal{N}$ that pointed down before t and up after t . For a spin on site j that undergoes multiple Raman processes, the same reasoning allows us to treat it as a field of strength $2J_{jk}/\mathcal{N}$ that acts for a period of time $\tau_j = (\tau_j^{\text{up}} - \tau_j^{\text{down}})$, where $\tau_j^{\text{up(down)}}$ is the total amount of time that spin j spends pointing up(down) along z (see Fig. 2). Hence we are free to evaluate all of the jump operators at $t = 0$ to give $|\tilde{\psi}\rangle = \prod_j \hat{\mathcal{Q}}_j |\psi\rangle$, thus ignoring the time-ordering in Eq. (5), so long as we evolve $|\tilde{\psi}\rangle$ with a modified time-evolution operator

$$\mathcal{U} = \exp \left[-it \left(\mathcal{H}' + \sum_j (\eta_j - i\gamma) \hat{\sigma}_j^z \right) \right]. \quad (6)$$

Here $\eta_j = \frac{1}{\mathcal{N}t} \sum_k J_{jk} \tau_k$ accounts for the magnetic field of all Raman-flipped ions, \mathcal{H}' is obtained from \mathcal{H} by ignoring the spin-spin couplings to spins that have undergone Raman jumps, and γ accounts for the non-Hermitian part of \mathcal{H}_{eff} that is *not* proportional to the identity operator.

The expectation value of an arbitrary operator $\hat{\mathcal{O}}$ at the end of a particular quantum trajectory is therefore given by $\langle \hat{\mathcal{O}} \rangle = \langle \tilde{\psi} | \mathcal{U}^\dagger \hat{\mathcal{O}} \mathcal{U} | \tilde{\psi} \rangle / \langle \tilde{\psi} | \mathcal{U}^\dagger \mathcal{U} | \tilde{\psi} \rangle$, and formally taking the average over all trajectories (denoted by an overbar) we have $\text{tr}[\rho \hat{\mathcal{O}}] = \overline{\langle \hat{\mathcal{O}} \rangle}$. Along with Eq. (6), these constitute a formal solution for the dynamics of any observable. We now proceed to derive closed-form expressions for the transverse spin-length and spin-spin correlation functions—which have not been derived previously even in the *absence* of decoherence. These are the central result of this paper.

Transverse-spin length and correlation functions.—Relaxation of the transverse spin-length in an Ising-type spin model is a canonical example of equilibration in a closed quantum system [30–32]. In the present model, such relaxation occurs due to a combination of the pro-

lification of quantum fluctuations *and* the equilibration with the environment (decoherence). Our theoretical treatment allows for both effects to be treated simultaneously, and therefore, in principle, the disentangling of these two physically different (but consequentially similar) processes. While such relaxation does not directly indicate quantum correlations or entanglement, in the absence of decoherence it nevertheless is entirely due to the buildup of quantum correlations—at the mean-field level coherent relaxation is absent. The correlations that develop during the dynamics can be understood in more detail by looking at two-spin correlation functions—for instance, these furnish a complete description of spin squeezing [33]. In the absence of decoherence and for all-to-all coupling ($\zeta = 0$), it is well known [33] that the transverse spin component revives at a time $\tau_r = \mathcal{N}\pi\hbar/(2J)$, and that a highly-entangled macroscopic-superposition state (MSS) appears at time $\tau_r/2$. This state is characterized by vanishing spin length but maximum transverse spin fluctuations, and our solution can be used to assess the robustness of such fluctuations against decoherence.

To calculate the transverse spin-length along a particular trajectory, we assign the discrete-valued variables \mathcal{R}_j and \mathcal{F}_j to each lattice site, which count the number of Raman jumps and Rayleigh jumps, respectively. As we have discussed earlier, all jump operators can be applied at $t = 0$, and therefore specifying the $\{\mathcal{R}_j, \mathcal{F}_j, \tau_j\}$ on each site fully determines transverse spin-length along that trajectory. The transverse spin component in direction φ and on site j is given simply in terms of the spin-raising operator on that site, $\langle S_j^x \rangle = \cos \varphi \langle S_j^x \rangle + \sin \varphi \langle S_j^y \rangle = \text{Re}[e^{-i\varphi} \langle \hat{\sigma}_j^+ \rangle]$, and generalizing solutions obtained in Ref. [30, 34] we find,

$$\langle \hat{\sigma}_j^+ \rangle = \frac{\alpha_j \beta_j \sin \theta_j e^{i\varphi_j}}{2g_j(2\gamma t)} \prod_{k \neq j} e^{\frac{2iJ_{jk}\tau_k}{\mathcal{N}}} \frac{g_k[2\alpha_k t(\gamma - iJ_{jk}/\mathcal{N})]}{g_k(2\gamma t\alpha_k)}. \quad (7)$$

Here $g_j(x) = \sum_{\sigma} |f_j(\sigma)|^2 e^{-\sigma x}$, $\alpha_j = \delta_{\mathcal{R}_j, 0}$ (δ being the Kronecker-delta symbol), $\beta_j = (-1)^{\mathcal{F}_j}$, and the details of the calculation are given in the Supplementary material. Defining a function $\mathcal{P}(\mathcal{R}, \mathcal{F}, \tau)$ that determines the probability distribution of these variables on a given lattice site, we have

$$\overline{\langle \hat{\sigma}_j^+ \rangle} = \sum_{\text{all } \mathcal{R}} \sum_{\text{all } \mathcal{F}} \int \dots \int \prod_k d\tau_k \mathcal{P}(\mathcal{R}_k, \mathcal{F}_k, \tau_k) \langle \hat{\sigma}_j^+ \rangle. \quad (8)$$

Equation (8) constitutes a formal solution for $\overline{\langle \hat{\sigma}_j^+ \rangle}$, and it can always be evaluated efficiently by averaging $\langle \hat{\sigma}_j^+ \rangle$ over stochastically generated trajectories. However, because the noise is uncorrelated from site to site, and accordingly the expression inside the product of Eq. (7) depends only on local stochastic variables, these sums and integrals factor into \mathcal{N} independent sets (each over the three stochastic variables), admitting closed form ex-

pressions. At this point, to avoid unnecessary complications in the ensuing expressions, we will take our initial state to point along the x axis ($\theta = \pi/2$, $\varphi = 0$), but our results easily generalize [35]. In the Supplement we explain how to evaluate these sums and integrals, and here we simply quote the result. Defining

$$\Phi(J, t) = e^{-\lambda t} \left[\cos(t\sqrt{s^2 - r}) + \lambda t \text{sinc}(t\sqrt{s^2 - r}) \right],$$

with $\lambda = \Gamma_r/2$, $s = 2i\gamma + 2J/\mathcal{N}$ and $r = \Gamma_{\text{ud}}\Gamma_{\text{du}}$, we find

$$\overline{\langle \hat{\sigma}_j^+ \rangle} = \frac{1}{2} e^{-\Gamma t} \prod_{k \neq j} \Phi(J_{jk}, t), \quad (9)$$

where the total decoherence rate is defined $\Gamma = \frac{1}{2}(\Gamma_r + \Gamma_{\text{el}})$ [38]. Similar calculations to those described above yield spin-spin correlation functions

$$\overline{\langle \hat{\sigma}_j^\mu \hat{\sigma}_k^\nu \rangle} = \frac{1}{4} e^{-2\Gamma t} \prod_{l \notin \{j, k\}} \Phi(\mu J_{jl} + \nu J_{kl}, t) \quad (10)$$

$$\overline{\langle \hat{\sigma}_j^\mu \hat{\sigma}_k^z \rangle} = \frac{1}{2} e^{-\Gamma t} \Psi(\mu J_{jk}, t) \prod_{l \notin \{j, k\}} \Phi(\mu J_{jl}, t), \quad (11)$$

with $\Psi(J, t) = e^{-\lambda t} (is - 2\gamma)t \text{sinc}(t\sqrt{s^2 - r})$ and $\mu, \nu = \pm$. These correlation functions, along with similar ones obtained by interchange of the site indices, completely determine the spin-spin correlations. Each instance of $\hat{\sigma}^x$ or $\hat{\sigma}^y$ in an observable generates an overall multiplicative factor of $e^{-\Gamma t}$; this is the effect of decoherence at the single-particle level. The structure of $\Phi(J, t)$ captures the effect of decoherence on the many-body physics, and could not have been deduced without our exact treatment.

We note that for $\gamma = 0$ the function $\Phi(J, t)$ undergoes a qualitative transition from oscillatory ($s^2 > r$) to damped ($s^2 < r$) behavior when $\Gamma_r = \Gamma_r^c \equiv 4J/\mathcal{N}$ [Fig. 3(a)]. Therefore, when there is only one coupling strength J , for instance in the case of all-to-all or nearest-neighbor couplings, dynamics of the transverse spin-length and correlation functions undergoes the same transition. More generally, for nonzero γ and any ζ , correlation functions exhibit a series of Hopf bifurcations as Γ_r is tuned. We note that the factors of $e^{-\Gamma t}$ in Eqs. (9-11) may make this transition difficult to observe experimentally, since the correlations are rapidly suppressed at the critical Γ_r^c .

Application to a trapped ion quantum simulator.— Trapped ion systems can simulate the Hamiltonian in Eq. (1), and can accurately measure the decoherence rates Γ_{el} , Γ_{ud} , and Γ_{du} . We note that both the Born-Markov approximation and the assumption of uncorrelated decoherence processes are extremely well justified for trapped ion systems [28]. Sample averaged spin-length and spin-spin correlation functions are easily measured in these experiments by looking at the length (and its shot-to-shot fluctuations) of various projections of the Bloch vector.

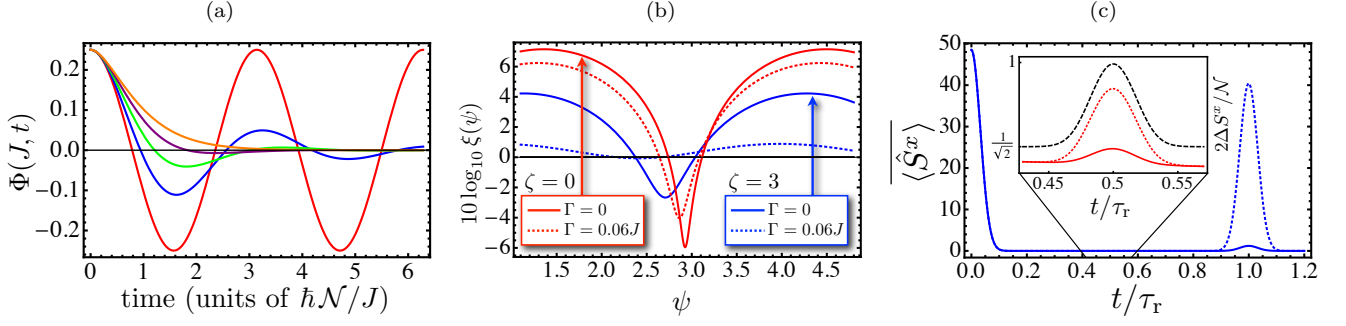


FIG. 3: (Color online) (a) Plots of $\Phi(J, t)$ for $\gamma = 0$ and $\Gamma_r/\Gamma_r^c \in \{0, 1/4, 1/2, 3/4, 1\}$, showing a transition from oscillatory to damped behavior. (b) Example of how spin squeezing is affected by decoherence (dashed lines are $\Gamma = 0.06J$, solid lines are for $\Gamma = 0$) for long-ranged (red, $\zeta = 0$) and short-ranged (blue, $\zeta = 3$) interactions. (c) Transverse spin relaxation and revivals for $\zeta = 0$, with parameters corresponding to expected experimental capabilities in [15] (blue solid line). The dotted blue line is obtained by treating decoherence at the single-particle level, and underestimates the detrimental effect of Raman decoherence by about a factor of 35. Inset: transverse spin fluctuations peaking at time $\tau_r/2$. Experimental parameters, exact treatment (red solid line); experimental parameters, single-particle treatment (red-dotted line); no decoherence (black dashed line).

In the trapped ion experiments discussed in Ref. [15], when $\zeta = 0$ the time scale at which quantum correlations become important for these observables, τ_c , scales with some power of the ion number: $\tau_c \sim \mathcal{N}^{1/3}$ for spin-squeezing, $\sim \mathcal{N}^{1/2}$ for transverse-spin relaxation, and $\sim \mathcal{N}$ for the creation of MSS's. Taking $\mathcal{N} = 100$ and $\Gamma = 0.06J$, as is typical in that experiment, we expect the proper incorporation of decoherence to be quantitatively important even for spin-squeezing, despite it being a relatively short-time indication of entanglement.

Equations (10,11) allow us to exactly calculate the effect of decoherence and the finite range of interactions on the maximum spin squeezing achievable in experiment. Figure 3(b) shows the expected squeezing and anti-squeezing as a function of angle ψ in the yz plane for $\zeta = 0, 3$, with $\Gamma = 0.06J$ and $\Gamma_{el} = 8\Gamma_{ud} = 8\Gamma_{du}$ (typical experimental numbers in [15]). The effects of decoherence are more pronounced for shorter-range interactions due to the longer time scales for maximal squeezing. For this calculation the spins are assumed to be initialized (prior to the Ising dynamics) in a coherent state pointing along the x -axis, and we define the spin squeezing parameter

$$\xi(\psi) = \frac{\sqrt{\mathcal{N}\Delta S^\psi}}{\langle \hat{S}^x \rangle}, \quad (12)$$

with $\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$, $\hat{S}^\psi = \frac{1}{2} \sum_j \hat{\sigma}_j^\psi$, and $\hat{\sigma}_j^\psi = \sin(\psi)\hat{\sigma}_j^y + \cos(\psi)\hat{\sigma}_j^z$. In Fig. 3(b), ξ has been calculated at the time when maximal squeezing occurs.

For the current experimental parameters there is essentially no spin revival, and no indication of a MSS at $\tau_r/2$. However, assuming 97 ions and expected improvements in the experiment [15] (a roughly 50-fold increase in the ratio J/Γ), in Fig. 3(c) we show that transverse spin revivals begin to appear. In that figure, the dot-

ted line is obtained by treating the decoherence at the single-particle level, which amounts to attaching a decaying exponential $e^{-\Gamma t}$ to the operators $\hat{\sigma}^x$ and $\hat{\sigma}^y$. We note that this treatment would be exact if the decoherence were only of the Rayleigh type ($\Gamma_r = 0$). The solid line is the full solution from Eq. (9); the large (~ 35 -fold) discrepancy between the single-particle and exact results indicates that our theory will be essential for understanding the behavior of this experiment. In the inset of Fig. 3(c) we plot the transverse spin fluctuations ΔS^x at times near $\tau_r/2$. The peak in these fluctuations (which would achieve unity in the absence of decoherence) is a result of strong transverse-spin correlations in an emerging MSS, indicating that the expected improvements to the experiments will bring within reach the production of MSS's of ~ 100 ions in the near future.

Conclusions.—We have exactly solved the non-equilibrium dynamics of arbitrary Ising models in the presence of single-particle decoherence. These calculations provide a rare glimpse into the exact structure of relaxation dynamics in open and strongly interacting quantum systems. They will allow for quantitative insights into experiments studying interacting spin-systems that are not perfectly isolated from the environment. In particular, they provide a rigorous foundation for benchmarking trapped-ion quantum simulators when interactions are strong and decoherence is non-negligible on experimental timescales.

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taneous calculation of spin-spin correlation functions for quantum Ising models in the absence of decoherence [36]. This manuscript is the contribution of NIST and is not subject to U.S. copyright.

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 - [37] In its current form it can describe, for instance, coupling of the spins to a thermal bath. The master equation could be slightly generalized by allowing the decoherence rates to vary from site to site. This can be taken into account trivially by adding site indices to these rates in the final expressions, Eqs. (9-11).
 - [38] Despite superficial appearances, notice that $\Phi(0, t) = 1$ (and not $e^{-\Gamma t/2}$), justifying the designation of Γ as the total decay rate, as in [28].

Expressions for spin length and correlation functions along a single trajectory

As in the main text, we write the state of a single spin as $\sum_{\sigma_j^z} f_j(\sigma_j^z) |\sigma_j^z\rangle$, where σ_j^z is an index that takes on the eigenvalues of the operator $\hat{\sigma}_j^z$, $f_j(1) = \cos(\theta_j/2) e^{i\varphi_j/2}$, and $f_j(-1) = \sin(\theta_j/2) e^{-i\varphi_j/2}$. The initial state of the entire system is taken to be a direct product of states for each individual spin

$$\begin{aligned}
 |\psi(0)\rangle &\equiv \bigotimes_j \sum_{\sigma_j} f_j(\sigma_j) |\sigma_j\rangle \\
 &= \sum_{\sigma_1^z, \dots, \sigma_N^z} f_1(\sigma_1^z) \times \dots \times f_N(\sigma_N^z) |\sigma_1^z, \dots, \sigma_N^z\rangle.
 \end{aligned} \tag{S1}$$

As discussed in the main text, evaluating all of the Rayleigh jumps at $t = 0$ can be accomplished by changing $\varphi_j \rightarrow \varphi_j + \pi \mathcal{F}_j$, which rotates the spin on site j by an angle π if it has undergone an odd number of Rayleigh jumps. Raman jumps can be incorporated at $t = 0$ by setting $\theta_j = 0(\pi)$ if the final Raman jump on site j was $\hat{\sigma}_j^+(\hat{\sigma}_j^-)$. We therefore define $\tilde{f}_j(\sigma_j^z)$ to be the modification of $f_j(\sigma_j^z)$ under those transformations. We also note that spins having undergone one or more Raman jumps are treated as an effective magnetic field, and not included in the spin-spin coupling term of the Hamiltonian. This is accomplished by changing $J_{ij} \rightarrow \alpha_i \alpha_j J_{ij}$ (with $\alpha_j = \delta_{\mathcal{R}_j, 0}$) in the Hamiltonian and including an effective magnetic field. Therefore, we can write the (time-dependent) wavefunction $|\tilde{\psi}(t)\rangle$ evolving under \mathcal{U} [as defined in Eq. (6) of the manuscript] as

$$|\tilde{\psi}(t)\rangle = \sum_{\sigma_1^z, \dots, \sigma_N^z} \exp \left[-it \left(\frac{1}{N} \sum_{i>j} \alpha_i \alpha_j J_{ij} \sigma_i^z \sigma_j^z + \sum_j (\eta_j - i\gamma) \sigma_j^z \right) \right] \tilde{f}_1(\sigma_1^z) \times \dots \times \tilde{f}_N(\sigma_N^z) |\sigma_1^z, \dots, \sigma_N^z\rangle. \quad (\text{S2})$$

In order to calculate the transverse spin-length we will first calculate $\langle \hat{\sigma}_j^+ \rangle = \langle \tilde{\psi}(t) | \hat{\sigma}_j^+ | \tilde{\psi}(t) \rangle / \langle \tilde{\psi}(t) | \tilde{\psi}(t) \rangle$, and at the end obtain $\langle \hat{S}^x \rangle = \text{Re} \left(\sum_j \langle \hat{\sigma}_j^+ \rangle \right)$.

Let's imagine, in particular, calculating $\langle \tilde{\psi}(t) | \hat{\sigma}_1^+ | \tilde{\psi}(t) \rangle$ (there is nothing special about the first spin, this just makes the notation in what follows less confusing). Because the wavefunction enters twice, this would involve two sums like the one in Eq. (S2), over σ_j^z and $\sigma_j'^z$, but very few terms survive: We need $\sigma_1^z = -1$, $\sigma_1'^z = 1$, and for all $j \neq 1$ we must have $\sigma_j^z = \sigma_j'^z$, so the matrix element is given by

$$\langle \tilde{\psi}(t) | \hat{\sigma}_1^+ | \tilde{\psi}(t) \rangle = \tilde{f}_1^*(1) \tilde{f}_1(-1) \sum_{\sigma_2^z, \dots, \sigma_N^z} |\tilde{f}_2(\sigma_2^z)|^2 \times \dots \times |\tilde{f}_N(\sigma_N^z)|^2 \exp \left[2it \left(\eta_1 + \sum_{j=2}^N \frac{1}{N} \alpha_1 \alpha_j J_{1j} \sigma_j^z + i \alpha_j \gamma \sigma_j^z \right) \right]. \quad (\text{S3})$$

If $\alpha_j = 0$, the j^{th} spin always has a well defined value of σ_j^z and the choice to include the term $\gamma \sigma_j^z$ in the exponentiated sum (or not) affects only the overall normalization of the wavefunction. By multiplying γ by α_j (in the exponentiated sum), we have chosen to not include the term $\gamma \sigma_j^z$, and this is properly accounted for when normalizing this expectation value below. In order to obtain $\langle \hat{\sigma}_1^+ \rangle$ we must divide by the (non-conserved) normalization of the wavefunction $\langle \tilde{\psi}(t) | \tilde{\psi}(t) \rangle$. Defining $g_j(x) = \sum_{\sigma} |f_j(\sigma)|^2 e^{-\sigma x}$ (as in the manuscript), we obtain

$$\langle \hat{\sigma}_1^+ \rangle = \frac{\tilde{f}_1^*(1) \tilde{f}_1(-1)}{g_1(2\gamma t)} \sum_{\sigma_2^z, \dots, \sigma_N^z} \frac{|\tilde{f}_2(\sigma_2^z)|^2}{g_2(2\alpha_2 \gamma t)} \times \dots \times \frac{|\tilde{f}_N(\sigma_N^z)|^2}{g_N(2\alpha_N \gamma t)} \exp \left[2it \left(\eta_1 + \sum_{j=2}^N \frac{1}{N} \alpha_1 \alpha_j J_{1j} \sigma_j^z + i \alpha_j \gamma \sigma_j^z \right) \right]. \quad (\text{S4})$$

This expression can be simplified by making the following set of observations: (1) $\tilde{f}_1^*(1) \tilde{f}_1(-1) = \alpha_1 \beta_1 e^{i\varphi_1} \sin(\theta_1)/2$ [as in the text, $\beta_j = (-1)^{\mathcal{F}_j}$], (2) the α_1 in the exponent is irrelevant, because if it takes the value 0 the entire expression for $\langle \tilde{\psi}(t) | \hat{\sigma}_1^+ | \tilde{\psi}(t) \rangle$ vanishes, and (3) the summand factorizes into a product where each term contains only local (i.e. on a single site) variables, and hence the sum of products can be exchanged for a product of sums. Taking (1-3) into account we obtain

$$\langle \hat{\sigma}_1^+ \rangle = \frac{\sin(\theta_1) e^{i\varphi_1}}{2g_1(2\gamma t)} \alpha_1 \beta_1 \prod_{j=2}^N \left(\sum_{\sigma_j^z} \frac{|f_j(\sigma_j^z)|^2}{g_j(2\alpha_j \gamma t)} \exp [2it \alpha_j \sigma_j^z (J_{1j}/N + i\gamma)] e^{2iJ_{1j}\tau_j/N} \right). \quad (\text{S5})$$

$$= \frac{\sin(\theta_1) e^{i\varphi_1}}{2g_1(2\gamma t)} \alpha_1 \beta_1 \prod_{j=2}^N \left(e^{2iJ_{1j}\tau_j/N} \frac{g_j[2\alpha_j t(\gamma - iJ_{1j}/N)]}{g_j(2\gamma t \alpha_j)} \right). \quad (\text{S6})$$

Therefore, we can write

$$\langle \hat{S}^x \rangle = \Re \sum_{j=1}^N \left[\frac{\sin(\theta_j) e^{i\varphi_j}}{2g_j(2\gamma t)} \alpha_j \beta_j \prod_{k \neq j} \left(e^{2iJ_{jk}\tau_k/N} \frac{g_k[2\alpha_k t(\gamma - iJ_{jk}/N)]}{g_k(2\gamma t \alpha_k)} \right) \right]. \quad (\text{S7})$$

The calculation of correlation functions follows from extremely similar considerations. For instance, let's consider $\mathcal{G}_{jk}^{-+} \equiv \langle \hat{\sigma}_j^- \hat{\sigma}_k^+ \rangle$. In this case, the operators in the expectation value only connect two states if $-\sigma_j'^z = \sigma_j^z = 1$, $\sigma_k^z = -\sigma_k'^z = 1$, and $\sigma_l'^z = \sigma_l^z$ whenever $l \notin \{j, k\}$. Therefore, much as before we have

$$\mathcal{G}_{jk}^{-+} = \frac{\sin(\theta_j) \sin(\theta_k) e^{i(\varphi_k - \varphi_j)}}{4g_j(2\gamma t) g_k(2\gamma t)} \alpha_j \alpha_k \beta_j \beta_k \prod_{l \notin \{j, k\}} \left(e^{2i(J_{kl} - J_{jl})\tau_l/N} \frac{g_l[2\alpha_l t(\gamma - i[J_{kl} - J_{jl}]/N)]}{g_l(2\gamma t \alpha_l)} \right). \quad (\text{S8})$$

Computing correlation functions involving a single $\hat{\sigma}^z$, such as $\mathcal{G}_{jk}^{z+} \equiv \langle \hat{\sigma}_j^z \hat{\sigma}_k^+ \rangle$ can be achieved by inserting σ_j^z into the sum in Eq. (S4), yielding

$$\mathcal{G}_{jk}^{z+} = \frac{\sin(\theta_k) e^{i\varphi_k} \alpha_k \beta_k}{2g_k(2\gamma t)} \left[\alpha_j \frac{\cos^2(\theta/2) e^{-2\gamma t} - \sin^2(\theta/2) e^{2\gamma t}}{g_j(2\gamma t)} + (1 - \alpha_j) \kappa_j \right] \prod_{l \notin \{j,k\}} \left(e^{2iJ_{kl}\tau_l/\mathcal{N}} \frac{g_l[2\alpha_l t(\gamma - iJ_{kl}/\mathcal{N})]}{g_l(2\gamma t\alpha_l)} \right). \quad (\text{S9})$$

Assuming one or more Raman flip occurred, the variable κ_j takes on the values ± 1 if the final Raman jump is $\hat{\sigma}_j^\pm$.

Analytic evaluation of stochastic averaging of trajectories

At this point in the calculation, for clarity of presentation, we set $\varphi_j = 0$ and $\theta_j = \pi/2$ (for all j), so all spins point along the x axis at $t = 0$. Defining $\mathcal{P}(\mathcal{R}, \mathcal{F}, \tau)$ to be the probability distribution of the variables \mathcal{R} , \mathcal{F} , and τ on a single site (it is the same on every site), the trajectory averaged expectation value is given by

$$\overline{\langle \hat{\sigma}_j^+ \rangle} = \sum_{\text{all } \mathcal{R}} \sum_{\text{all } \mathcal{F}} \int d\tau_1 \dots \int d\tau_{\mathcal{N}} \langle \hat{\sigma}_j^+ \rangle \prod_k \mathcal{P}(\mathcal{R}_k, \mathcal{F}_k, \tau_k). \quad (\text{S10})$$

To begin, we note that the probability distribution can be decomposed as $\mathcal{P}(\mathcal{R}, \mathcal{F}, \tau) = \mathcal{P}_{\text{el}}(\mathcal{F}) \mathcal{P}_{\text{r}}(\mathcal{R}, \tau)$, which is valid because the probability of Rayleigh jump is independent of whether a Raman jump has occurred (and vice versa). The occurrence of random processes follows a Poissonian distribution, so $\mathcal{P}_{\text{el}}(\mathcal{F}) = e^{-\Gamma_{\text{el}} t/4} (\Gamma_{\text{el}} t/4)^{\mathcal{F}} / \mathcal{F}!$, and we have also calculated \mathcal{P}_{r} . The result depends on whether \mathcal{R} is even or odd (the proof is simple but requires some careful reasoning, and we do not give it here). We parameterize the \mathcal{R} -odd solution by $\mu = (\mathcal{R} - 1)/2$ (which will run over all non-negative integers), and we parameterize the \mathcal{R} -even solutions by $\mu = (\mathcal{R} - 2)/2$ (once again running μ over all nonnegative integers), and obtain

$$\mathcal{P}_{\text{r}}^{\text{odd}}(\mu, \tau) = \frac{\Gamma_{\text{r}}}{4} e^{-\Gamma_{\text{r}} t/2} \frac{(\Gamma_{\text{ud}} \Gamma_{\text{du}}/4)^{\mu}}{(\mu!)^2} e^{-2\tau\gamma} (t^2 - \tau^2)^{\mu} \quad (\text{S11})$$

$$\mathcal{P}_{\text{r}}^{\text{even}}(\mu, \tau) = \frac{\Gamma_{\text{ud}} \Gamma_{\text{du}} t}{4} e^{-\Gamma_{\text{r}} t/2} \frac{(\Gamma_{\text{ud}} \Gamma_{\text{du}}/4)^{\mu}}{\mu! (\mu + 1)!} e^{-2\tau\gamma} (t^2 - \tau^2)^{\mu}. \quad (\text{S12})$$

By evaluating the sum

$$\sum_{\mathcal{F}=0}^{\infty} \mathcal{P}_{\text{el}}(\mathcal{F}) e^{i\pi\mathcal{F}} = e^{-\Gamma_{\text{el}} t/2}, \quad (\text{S13})$$

we obtain

$$\begin{aligned} \overline{\langle \hat{\sigma}_j^+ \rangle} &= \frac{e^{-\Gamma_{\text{el}} t/2}}{2 \cosh(2\gamma t)} \sum_{\mathcal{R}_1, \dots, \mathcal{R}_{\mathcal{N}}} \int d\tau_1 \dots \int d\tau_{\mathcal{N}} \left[\mathcal{P}(\mathcal{R}_j, \tau_j) \alpha_j \prod_{k \neq j} e^{2iJ_{jk}\tau_k/\mathcal{N}} \frac{\cosh[2\alpha_k t(\gamma - iJ_{jk}/\mathcal{N})]}{\cosh(2\gamma t\alpha_k)} \mathcal{P}_{\text{r}}(\mathcal{R}_k, \tau_k) \right] \\ &= \frac{e^{-\Gamma_{\text{el}} t/2}}{2 \cosh(2\gamma t)} \left[\sum_{\mathcal{R}_j} \int d\tau_j \alpha_j \mathcal{P}(\mathcal{R}_j, \tau_j) \right] \times \left[\prod_{j \neq k} \sum_{\mathcal{R}_k} \int d\tau_k e^{2iJ_{jk}\tau_k/\mathcal{N}} \frac{\cosh[2\alpha_k t(\gamma - iJ_{jk}/\mathcal{N})]}{\cosh(2\gamma t\alpha_k)} \mathcal{P}_{\text{r}}(\mathcal{R}_k, \tau_k) \right]. \end{aligned}$$

Because α_j gives 1 if there have not been any Raman flips at site j and 0 otherwise, the term in the first square bracket is just the probability that there has been no Raman flip on site j , which is given by $e^{-\Gamma_{\text{r}} t/2} \cosh(2\gamma t)$ [this comes from evolving the wavefunction of a single spin pointing along x with the effective Hamiltonian Eq. (4)]. Therefore we have

$$\overline{\langle \hat{\sigma}_j^+ \rangle} = \frac{1}{2} e^{-\Gamma t} \times \left[\prod_{j \neq k} \sum_{\mathcal{R}_k} \int d\tau_k e^{2iJ_{jk}\tau_k/\mathcal{N}} \frac{\cosh[2\alpha_k t(\gamma - iJ_{jk}/\mathcal{N})]}{\cosh(2\gamma t\alpha_k)} \mathcal{P}_{\text{r}}(\mathcal{R}_k, \tau_k) \right] \quad (\text{S14})$$

Defining $s = 2i\gamma + 2J/\mathcal{N}$, the trick is now to evaluate the quantity

$$\begin{aligned} \Phi(J, t) &= \sum_{\mathcal{R}} \int d\tau \mathcal{P}(\mathcal{R}, \tau) e^{2iJ\tau/\mathcal{N}} \frac{\cosh(ist\alpha)}{\cosh(2\gamma t\alpha)} \\ &= e^{-\Gamma_{\text{r}} t/2} \cosh(ist) + \sum_{\mathcal{R}=1}^{\infty} \int d\tau \mathcal{P}(\mathcal{R}, \tau) e^{2iJ\tau/\mathcal{N}}. \end{aligned} \quad (\text{S15})$$

The second equality follows from pulling off the $\mathcal{R} = 0$ term in the sum, which represents the probability of having no Raman flip (and so we can set $\tau = 0$ and $\alpha = 1$ in this term). The second term represents the probability for any finite number of Raman flips, and hence we must keep τ arbitrary but can set $\alpha = 0$. The integral over τ can be evaluated by using the identity

$$\int_{-t}^t d\tau (t^2 - \tau^2)^\mu e^{-ix\tau} = (2t)^{\mu+1} \frac{j_\mu(xt) \mu!}{(x)^\mu}, \quad (\text{S16})$$

where j is a spherical Bessel function. The remaining sum over \mathcal{R} can be recognized as a generating function for the spherical Bessel functions (or a derivative thereof). Defining parameters $\lambda = \Gamma_r/2$ and $r = \Gamma_{ud}\Gamma_{du}$, and functions

$$F(x, y) = \text{sinc}(\sqrt{x^2 - y}) \quad (\text{S17})$$

$$G(x, y) = \frac{\cos(\sqrt{x^2 - y}) - \cos(x)}{x}, \quad (\text{S18})$$

we obtain

$$\Phi(J, t) = e^{-\lambda t} \cos(st) + \lambda t e^{-\lambda t} F(st, rt) + s t e^{-\lambda t} G(st, rt) \quad (\text{S19})$$

$$= e^{-\lambda t} \left[\cos\left(t\sqrt{s^2 - r}\right) + \lambda t \text{sinc}\left(t\sqrt{s^2 - r}\right) \right] \quad (\text{S20})$$

We can now write out the exact solution

$$\overline{\langle \hat{S}^x \rangle} = \frac{e^{-\Gamma t}}{2} \text{Re} \sum_j \prod_{k \neq j} \Phi(J_{jk}, t). \quad (\text{S21})$$

Because Eqs. (S7,S8) have such a similar structure, the stochastic averaging of correlation functions is almost identical, and leads to the similar expressions given in the main text [Eqs. (10,11)].
